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Chapter 9 Summary and Conclusions

This chapter concludes the thesis. It sets off by reviewing the main findings and associating them with the three aims defined in Chapter Three. These are followed by notes on the limitations of this study, directions for future research and some final comments.

9.1 Overview

Chapter One began with a story of spontaneous learning, and asked how we could design for such events to occur. This simple question was the seed for an enquiry into the nature of education as a science of designed learning, with specific attention to technology enhanced mathematics education (TEME). The characterisation of education as designed learning established a multi-faceted link between design and epistemology, or the creation of knowledge. This link is a motif that weaves through this thesis. Design emerged as a method of study, an object of study and an outcome of study, leading to an overarching theme of this thesis, to:

Consider the study of learning as a design science. Highlight the implications of such a paradigm, and propose ways to theorise design in a manner which draws both on educational research and computer science.

The pragmatist nature of a design stance to educational research suggests a tight dependency between the three levels of epistemology: the method by which we study education (its normative epistemology) needs to link our understanding of how people learn (genetic epistemology) to how we design artefacts for learning (generative epistemology). The specific aims of this thesis are derived from this realisation, to:

- Identify potential elements of an **epistemic infrastructure** for a design science of TEME.
- Combine and elaborate the elements identified into a coherent **methodological framework** in a given research TEME context.
- Apply the methodology in a problem domain and demonstrate its potential by producing a contribution towards a language of **pedagogical patterns** for TEME.

Sections 9.2, 9.3 and 9.4 relate the findings of this study per each one of these aims. Section 9.5 notes some of its limitations, and section 9.6 considers a few questions for future research.

9.2 Findings related to Aim 1: to identify potential elements of an epistemic infrastructure for a design science of TEME

The initial motivation for a design science perspective on TEME is the observation that the study of education is distinguished from other fields concerned with human learning in its attention to

designed learning. Where other sciences ask how and what humans learn, the science of education asks what they should learn and how best to support them to this end. This argument was expanded and supported by reference to the literature, in particular to Herbert Simon's seminal work. Simon (1969) distinguishes between the natural sciences and the sciences of the artificial, the sciences of design: natural science asks what *is*, the science of design asks what *ought to be*. A design science of learning needs to be informed by its natural science siblings such as biology, psychology and computational learning. It needs to integrate these bodies of knowledge and establish its own agenda, methods and language. A review of existing traditions of design-based research in TEME identified several critical gaps in the field's epistemological infrastructure.

Following Simon, four traits of design science were identified: a value-driven agenda, a functional axis of decomposition, the role of representation and an inherently iterative method of inquiry. All these are fundamental to the study of learning. The value dimension is reflected in education's remit to improve the life of individuals and communities. A functional axis of decomposition means that the objects of educational research should be categorized and analysed according to their effect rather than their structure. The role of representation is generally accepted in the learning sciences. Iterative methodologies are gaining ground, in particular where technology is involved.

A design approach would be better adapted to the complex and dynamic nature of the circumstances and questions studied by educational science. Furthermore, this approach has the potential to offer a cohesive paradigm, bridging across practice and multiple theories. These advantages are made even more salient in the face of the rapid pace of change induced by technological developments, calling for agile, responsive and proactive approaches to educational research.

Science strives to produce knowledge, and a design science should produce design knowledge.

Chapter Two characterised design knowledge in TEME as:

- **Problem driven, solution oriented, value laden:** design knowledge always departs from an unsatisfactory situation, i.e. a problem, and aims to move to a desired one by solving it. The ascription of desirability measures to world states is a matter of values.
- **Situated in context:** Problems and solutions are only valid with respect to a particular context, and that context needs to be articulated. Indeed, the field of TEME is often partitioned per context: primary vs. secondary or tertiary, formal or informal, etc.
- **Holistic (inherently inter-disciplinary):** A functional axis of decomposition cuts across structural distinctions. Problems are dissected into sub-problems, combining a variety of structural elements at all levels. Designing an educational instrument needs to take account of multiple dimensions regardless of its scale: pedagogical, social, affective, cognitive, cultural, technical, etc.

A review of the field identified the need for a clearly articulated consensual epistemic infrastructure; the explicit and implicit rules and assumptions which bound the discourse of the scientific community, and the a logical system by which claims are presented and justified, independently of their content. Some desirable features of such an infrastructure emerge from the discussion:

- **Accessibility;** arguments made by researchers should be readable by the scientific community, as well as practitioners, and policy makers.
- **Transparency and traceability;** the full cycle of a design study should be observable by an external reviewer, and most importantly the path that leads from theory to conjecture through experience and back to theory.
- **Expressiveness;** the forms used for communicating design research should allow for the articulation of all that is needed to support the above requirements.
- **Functional-pragmatist orientation;** the mechanisms used to organise and communicate knowledge in the design science of TEME need to be aligned with the nature of this knowledge. Given the pragmatist foundations and functional axis of design knowledge, the research community needs means for organising this knowledge accordingly.
- **Cumulativity;** the forms of presenting claims and arguments need to afford easy aggregation of knowledge, building new results on the basis of prior art.

Narrative was identified as a powerful epistemic form, leading to a proposal for a formalisation of design narrative as a form of scientific discourse. Design patterns emerged as a promising form for encoding design knowledge in educational research, but likewise, several challenges were identified. These challenges can be met by providing a rigorous methodological apparatus, which would include the detailed format of each representation and the procedures for transitioning between them, along with measurable criteria for validity.

The acknowledgment of education as designed learning calls for a design science perspective in educational research. Such an approach appears to be particularly well suited for TEME. A design science stance raises new questions and suggests new ways of answering them. Above all, it inspires a new agenda; of pragmatic research which contributed innovation to theory and practice as one.

9.3 Findings related to Aim 2: To combine and elaborate the elements identified into a coherent methodological framework in a given research context

Aim 2 is addressed by projecting the theoretical arguments and epistemological elements (Chapters Two and Four) onto the setting of the demonstrator research defined by Aim 3: “the design of tools and activities for learning about number sequences, in an extra-curricular lower-secondary school setting” (Chapter Three). The result is a concrete framework which offers a contribution towards a pattern-based methodology for a design science of TEME.

A review of existing design approaches to educational research exposed the emergence of shared practices and pockets of expertise (Chapter Four). Among the common methodological characteristics are a dual focus on practical and theoretical contributions, a highly interventionist and agile attitude, and a cycle of iterative research. This cycle includes phases of theory, design, implementation, execution (experiment / practice), articulation of experience, interpretation, evaluation and analysis, and feedback to both theory and design. The products of this cycle are validation or critique of existing theory, evidence regarding the effectiveness of artefacts and practices in well-defined settings, and innovations in practice and theory. A frequent by-product of research is the synthesis of multiple frameworks. This cycle is embedded in a meta-cycle, which includes a framing phase, an empirical phase and a retrospective analysis phase.

A need was identified for appropriate representations for expressing design knowledge in the various phases of these cycles, along with procedures for managing the transitions between phases. A particular innate trajectory from experience to knowledge was reviewed as a basis for these representations and transitions. This mechanism proceeds by narratisation of experience and genre transposition. Similar narratives are fused, thus abstracting similarities and eliminating detail. In this process temporal links are replaced by semantic relations.

By analogy to this model of innate learning processes, two representations were proposed: design narratives and design patterns. The former serves the interpretive phase of the design experiment cycle, in which the researcher organises the data and records the unfolding of events in the empirical phase. The latter serves the analytical and conjectural phases, allowing researchers to articulate situated abstractions of design knowledge derived from the experiment.

Once the research setting was described, it was used to illustrate how these constructs interacted in practice. Special attention was given to the analytical half of the design experiment cycle, which describes the trajectory from experience to structured theory, focusing on the processes by which data was collected and from it design narratives and design patterns were produced and validated. The result is a description of a methodological framework and set of instruments for the demonstrator study, from data collection and management, through interpretation and systemisation of observations as design narratives and on to the formalisation of research outcomes as design patterns. Taken together, they provide a full specification for implementation of the analytical hemicycle of the design experiment cycle and of the retrospective analysis phase of the design research meta-cycle.

The analytical hemi-cycle begins with the collection of data during the design, implementation and trial of educational innovations. Three classes of data were identified: design data, student

productions, and classroom observations. Design data include any record of the design process and its product. The challenges of a messy environment were addressed by redundancy, triangulation, and nearest substitute; collecting every available fragment of data, supporting claims by combining data from different sources, and identifying the pragmatically accessible forms of data closest to the ideal.

A structured process of selection and construction of design narratives was identified, using Bruner's ten principles as guidelines. These principles, adapted to the needs of scientific form, were expressed in the design narrative template.

Design patterns were extracted from design narratives through a six step process devised to capture the key design elements, systemise and substantiate them. This was followed by a phase of refactoring: structural manipulations which give the pattern language as a whole greater coherence.

The proof of this pudding is in its eating: regardless of its theoretical foundations and clarity of description, any methodology is only as good as the research it engenders. Thus, the outcomes of the demonstrator study reflect back on the framework and validate it.

Mathematical sciences use algebra as a language for communicating and preserving ideas. Likewise, a design science of TEME needs to identify languages suitable for its discourse. I find that pattern languages, integrating design narratives and design patterns, offer powerful tools in this respect. These representations of design knowledge fit well within the cycles of design research. Design narratives systematise an innate form of extracting knowledge from experience. They support the first tier of interpretation by affording rich contextualised descriptions of problem solving. Design patterns capture essential features across narratives, encapsulating recurring challenges and forces pertaining to a domain of learning design, the interactions between them and possible methods of solution. Together they fill a gap between theory and empirical evidence, and maintain their links with both.

9.4 Findings related to Aim 3: Apply the methodology in the problem domain and demonstrate its potential by producing a contribution towards a pattern language for technology enhanced mathematics education

The demonstrator study (Chapters Six to Eight) focused on designing activities and technology to support learning about number sequences through construction, communication and collaboration. It addressed Aim 3 by applying the methodological framework derived from Aim 1 in response to Aim 2.

The demonstrator study followed the design research cycle and meta-cycle (Chapter Four). The framing phase produced a pragmatic review of the literature (Chapter Six). The aim of this review was to identify key challenges in the domain and raise conjectures as to ways in which they may be addressed. These conjectures were translated into the design of activities and the technology to support them. The iterative process of design, implementation and evaluation was captured as a set of design narratives (Chapter Seven). These design narratives were analysed to produce the collection of design patterns (Chapter Eight).

Number patterns and sequences are broadly accepted as intuitively appealing to learners, and a viable gateway to advanced mathematical subjects, yet the review exposed several pedagogical and epistemic issues concerning this topic. Foremost among these are the difficulties of formulating a structural view of sequences, specifically with respect of the process-product duality and recursive vs. closed form. These difficulties were linked to a dissonance between intuitive perceptions and the prevalent school representations of mathematical objects. This observation was inspected in the broader context of the relationship between communication, representation and meaning in mathematical learning. Combining the communicational approach with the concept of situated abstraction led to considering narrative, as a fundamental social and cognitive epistemic force. Narrative form would appear to be at odds with the nature mathematics, raising the question of their reconciliation. Bringing narrative into the domain of constructionist learning and educational programming raises questions regarding its manifestation in computer-based representations, specifically, how to represent number sequences. The STREAMS design pattern was proposed as a promising candidate. The union of the various approaches calls for educational designs which weave construction, communication and collaboration.

The observations and conjectures emerging from the review of the literature were translated into design of activities and tools. These were tested and refined in two sites over three years. The process of design, and the outcomes of testing were captured in Chapter Seven as series of nine design narratives. These design narratives form the main bulk of empirical content in this thesis. They are the first tier of interpretation, not the data itself. Each narrative recounts a particular incident, defined by a single problem to be solved or task to be accomplished. An incident could span a single session, a few weeks or a few months.

The primary sources for design data are project reports, design documents, teacher manuals and research journals. The primary sources for student productions data are student webreports, ToonTalk code and paper-based written tasks. All texts and artefacts were read as mathematical arguments expressed in narrative. Acknowledging the impossibility of separating observation from intervention, data collection was integrated with activity design. Products were assessed in terms of

aptness, complexity and sophistication of argument. The primary sources for classroom observation data were field notes, video and audio recordings. Interview data included (individual and group) stimulated recall interviews, task-based interviews and in-activity probes. The latter played a central role in observational data.

Two types of design narratives emerged: researcher narratives (RNs) and learner narratives (LNs). RNs recount a pedagogical problem and its resolution from the researcher's point of view. The focus is on the integrated design and development of activities, social practices and supporting technology. LNs follow the learner as a designer, contending with a problem they encountered in the context of one of an activity, their use of the resources provided in confronting this problem, and the indications of their learning gains in the process. These are third person accounts based on the learners' written and verbal articulations and my observations. The two types of narratives are interdependent; the problems encountered by learners and their resolution are the drivers of their learning trajectory. The researcher's problem, from a bird's eye view, is to provide learners with an effective set of problems and the means for resolving them, so as to direct their learning trajectory. Thus, LNs illuminate and substantiate the RNs. Chapter Seven included six RNs and three LNs. These covered the three main subject themes and the supporting infrastructure, over the three years of design experiments. They were chosen to illustrate a variety of scale, data sources, and analytical foci, and to expose insights at different levels. Some are unique to the situation at hand, others generalise to a broader scope within TEME, and some reflect on the nature of design science, as practiced in this study.

Several generalisable observations stand out; the potential of the STREAMS design pattern, the importance of combining construction, communication and collaboration, the role of narrative, and the recursive intuition of sequences.

STREAMS proved apposite to teaching and learning about number sequences. It allowed learners to distinguish between process, parameters and product, construct complex mathematical entities from simple blocks, appreciate the characteristics and behaviour of classes of sequences, and articulate sophisticated arguments.

Activities were designed to encourage learners to assimilate mathematical concepts through a combination of construction, communication, and collaboration. The technological infrastructure evolved to support the streamlined integration of these modes of action. Construction challenges preconceptions and seeds new ideas, but these need reflection to be transformed to structured knowledge. Communication can drive reflection, but only if it can draw on substantial relevant experiences. Collaboration is a powerful motivator of sustained communication, and consequently

reflection. The trajectory from construction through communication to reflection highlights the importance of narrative in constructing mathematical knowledge, through guided transition from action to narrative to propositional discourse.

Several of the LNs demonstrate the recursive nature of the naïve concept of number sequences. Allowing learners to encode their intuitions in ToonTalk evinced how this concept is no less mathematical than the indexed view. Activities only gained momentum once their design acknowledged learners' intuitions and was adjusted to harness them, rather than try to fight them. Nevertheless, the initial iterations of design were necessary in order to expose those intuitions.

Among the more general themes that surface are the co-evolution of technology and pedagogy, the interdependence of interface and substance, and consequently the fluidity of design and the need for flexibility and malleability. This symbiosis between technology and epistemology was expressed in the fundamental, structural layers – but also at the level of the interface. The best ideas, expressed through the best technology, will fail if the interface by which the user interacts with the technology is not tuned to the underlying concepts and methods.

Finally, a lesson that shines through these design narratives is that no single element of design, technological or pedagogical, can be directly linked to a desired effect – only their careful assembly, adapted to the given context. This calls for means of analysing the narratives to identify and articulate individual elements of effective design, and then synthesising those to devise solutions for novel problems. This need is addressed by a collection of design patterns (Chapter Eight), contributing towards a pattern language of construction, communication and collaboration in technology enhanced mathematics education. This collection included seven fully specified patterns, and outlined another seven. They reflect an approach which sees the technological and pedagogical dimensions inseparable in the design of educational activities and tools. Thus, while some patterns emphasise certain features of technology and others highlight structures of activity, they all relate to some extent to both. The collection of patterns serves a dual purpose; for the educational designer it illuminates challenges which may arise in a given context, and some possible ways of dealing with them. For the research community, it opens the door for systematic scientific discourse of design for TEME.

The theme of combining construction, communication and collaboration was reified in patterns such as OBJECTS TO TALK WITH, BUILD THIS, CHALLENGE EXCHANGE, and POST LUDUS. This theme is tightly related to the issue of narrative, which manifests itself in patterns such as NARRATIVE SPACES and NARRATIVE REPRESENTATIONS.

GUESS MY X operationalises the principle of allowing formal concepts to flow from intuitions. The STREAM pattern captures naïve recursive intuitions of number sequences. Adapting it for educational settings produced TRANSPARENT STREAM, which provides functional modularity and prompts learners to acknowledge the relationship between process, parameters and product and to classify sequences by their behaviour.

Technology cannot engage learners with mathematical ideas without interfaces which provide access to these ideas. Such interfaces need to combine usability and pedagogy, as demonstrated by patterns such as SOFT SCAFFOLDING, TASK IN A BOX, SEMI-AUTOMATED META-DATA and ACTIVE WORKSHEETS.

Technology designed for learning needs to reflect fundamental educational principles and values. This ideal is exemplified by patterns such as MATHEMATICAL GAME PIECES and HARD BUT NOT TOO HARD. Both these patterns stem from two beliefs: first, that the origins of mathematical thinking are in a fundamental human desire to identify patterns in the world and explain them, and second, that the role of education is to nourish and guide this desire.

Above all, these patterns demonstrate the immense complexity of designing for learning, which calls for further efforts towards identifying methodical frameworks for describing, aggregating and mapping design knowledge. The prime example of this complexity is the GUESS MY X pattern. At first, guess my robot may seem a simple game with surprising effects. The detailed analysis embodied in the GUESS MY X pattern, along with its 'ancestor' patterns – such as CHALLENGE EXCHANGE and BUILD THIS, suggests that the game's success is not a happenstance, but rather a result of an intricate assemblage of multiple design elements relating to the tools, the activity and the ways in which they interact.

9.5 Limitations of this study

This thesis describes an arduous journey in uncharted waters. It would be of no surprise that upon arrival at my port of destination I can identify several junctions where in retrospect I would have taken a different path.

9.5.1 Evolution of the Methodological Framework

Picking up on the metaphor, the text of this thesis first obtains the necessary maps and constructs a sturdy methodological ship, and then follows her journey through the waters of the demonstrator study. Reality was closer to setting sail with a barge-full of charting paper, drafting maps, logging timber and constructing the ship en route. Had the methodological framework been available at the onset, the empirical work would have enjoyed a much smoother journey, and would have been

likely to produce more elaborate results. However, it was precisely the deficiency of a clear methodology that motivated this study, implying a secondary role for the demonstrator study. Undoubtedly, there are rich opportunities for exploring the demonstrator domain as a primary interest.

9.5.2 Further Validation of the Outcomes

The elements of an epistemic infrastructure described here, and the derived methodological framework, have been justified by theory and demonstrated in practice. However, these warrants still fall short of a robust validation. In order to stamp the epistemic and methodological constructs as scientifically credible they will need to be assessed by multiple researchers in varied circumstances, a requirement which transcends the nature of a PhD study.

9.5.3 Data collection

My experiments were conducted under 'messy' conditions: school classrooms where video and audio recording was often inefficient. The study of innovative learning technologies called for innovative data collection methods. These inherent constraints were amplified by my need to shift from a different paradigm and master the craft of educational research. Suitable methods were eventually identified and adapted to the specific settings of this study. However, valuable observations from the early iterations were lost due to missing or low-quality recordings.

9.5.4 Sensitivity to specific conditions

The availability of the WebReports and ToonTalk media, and the classroom practices which they inspired, created a social setting radically different from the typical classroom. In order to reduce the risks associated with testing innovative and potentially unstable technologies, experiment groups were small. Voluntary participation suggested prior interest and motivation. All these factors raise valid concerns regarding the possibility of replicating these experiments on a large scale with a randomly chosen population. However, the tools have been used by diverse groups across Europe, some larger in size, and the activities have been tested in other sites, although not in the same sequence. Nevertheless, all of these sites were under close attention of experienced researchers, which in turn had my dedicated support. While the epistemological and methodological arguments should extend beyond the specific context of this work, as should the derived design patterns, I wish to warn against any naïve attempt to extend the specific findings of the demonstrator study.

9.6 Open Questions and Future Research

This section concludes the chapter, and with it the thesis, by marking some directions for extending the work reported here. Mirroring the three aims of this thesis, the notes here refer to epistemology, methodology and pedagogy.

9.6.1 Extending the Epistemological Argument

The images of design knowledge and design science portrayed in this thesis are a view “from the inside out”; they originate in a study of TEME, and the recognition that this study is built on a fragile foundation. Consequently, the epistemological claims expressed here are confined to the domain of TEME. It would appear that they should be extendible to a broader remit, yet the radius of this remit and the effects of such an expansion are unclear. Should education, in general, be studied as a design science? How would such a perspective interact with others – such as the sociology, history or philosophy of education? How much of the edifice constructed here will survive such an expansion, and which parts are domain specific and will need to be remodelled?

The positions expressed here regarding the nature of a design science of TEME tread the border of philosophy of science and epistemology. Should they be expanded to a broader scope, this border needs to be explored and possibly crossed. The pragmatist foundation of design science and its implications need to be fully understood. Some current debates in these fields would appear to be relevant, for example the question of the relationship between knowledge-that and knowledge-how (Ryle, 1949; Stanley and Williamson, 2001), the relation between knowledge and actions (Hawthorne and Stanley, 2008), the dependence of knowledge on the practical consequences for the knower (Stanley, 2007) and the value dimension of scientific enquiry (Kitcher, 2001).

9.6.2 Extending the Methodological Framework

The design experiment cycle presented in Chapter Four was divided into two parts: the empirical hemicycle, flowing from theory to practice, and the analytical cycle feeding back from practice to theory. The methodological tools presented thereafter focused on the latter, while the former was treated more as a craft than as a scientific process. Cross (2001) notes that much of the effort in the design science tradition has focused on “scientising design”, devising scientifically valid manners of producing useful artefacts. By contrast, he proposes a view of design as a discipline, a reflective practice with its own culture and forms of knowledge. The questions that follows are: should this distinction be maintained? Has scientised design fallen from favour due to a fundamental flaw or a failure of implementation? If the science of design needs to be segregated from its craft, how should the boundaries and interchange channels be defined? If they can be blended, what are the scientific

tools suitable for the empirical arc of design research? It may be the case that previous attempts failed because they employed a positivist paradigm, incapable of responding to the dynamic complexity of design problems.

The power of narrative in constructing and communicating design knowledge is a theme that underlies much of this thesis. This power rests on an inherent paradox: narrative reflects on the general through the particular. The timeless is captured in the transient. A possible explanation would be that we use narrative to understand domains too complex for symbolic logic. Another option would be to argue for a systematic network of knowledge representations, linking the abstract and formalistic to the vernacular and indigenous.

The specific tools of data collection and analysis used in this study were limited to the scope of the demonstrator study, and should be expanded and woven into the emerging framework for design based TEME research. In particular, the media used by learners to express their ideas offer an extraordinary richness of representation. While some elements of multimodal analysis (Jewitt and Kress, 2003) were incorporated into the analysis, a more comprehensive examination is called for.

The methodological framework presented in this thesis is subject to a weakness, inherent to PhD work; a methodology is ultimately part of the norms of a scientific community. A PhD, on the other hand, is characterised as an individual contribution to science. This tension can only be resolved after the thesis has been submitted. It is my hope and belief that I have made a convincing argument in favour of the epistemic and methodological tools presented here. My next goal will be to share these tools with the TEME research community, and make them accessible to other researchers. I expect that, as they are adopted and appropriated by others, their lacunae will be exposed.

9.6.3 Extending the Pedagogical Findings

The communicational framework described in Chapter Six bears a striking affinity to models of computational discourse analysis and comprehension developed in the realm of artificial intelligence (Grosz & Sidner, 1986). The field of artificial intelligence in education (AIED) has developed a variety of tools for modelling learner discourse. The advantage of these tools is in their mathematical precision, required by any model that needs to be processed by a machine. Such models could be valuable as practical methodological tools. At the same time, as noted by Rickel et al (2001), they could be informed by the sophistication of socio-cultural theories.

Another avenue worth considering is a dynamical model for the relationship between symbols, meaning and learning. A recent trend in cognitive psychology views development from a dynamical systems perspective (Vallacher & Nowak, 1997; Beer, 2000; Lewis 2000).

The notion of programming as narrative form raises further questions. There is a notable difference between the narrative present in most of the literature and that expressed in programmed code. The former is predominantly a recount of past events, whereas the latter is a recipe for affecting future events. This distinction needs to be elaborated. In a way, programming is a form of fantasy: but perhaps so is mathematics? The narrative qualities of different programming environments should be inspected, and correlated to the nature of mathematical discourse.

The use of the STREAMS pattern raises the question of software design patterns as educational tools: design patterns might make it possible to provide participants in constructionist learning environments with the knowledge of how to create their tools, rather than use pre-fabricated tools. This should be an advantage where the tool embodies a mathematical idea in its very structure, or when understanding the workings of the tool is required for understanding its effects and uses. This conjecture calls for further research.

One theme that presents itself as a question for further research is the notion of learners as design scientists. Several parallels were drawn between my process of research, and students' learning trajectories, for example in comparing researcher narratives and learner narratives. These parallels should be further elaborated, and perhaps used explicitly as a basis for educational design.

9.7 *Final comments*

This chapter marks the endpoint of a long journey, but as with every journey – it is just the prologue to the next. I end this study with more questions opened than answered, and some answers to questions I have not considered at the onset. The study could only draw to a close when I accepted that it had limitations, as any work does, and some questions will need to be explored after it is sealed.